

Existence and uniqueness of the modified error function

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Abstract

This article is devoted to prove the existence and uniqueness of solution to the non-linear second order differential problem through which is defined the modified error function introduced in *Cho-Sunderland, J. Heat Transfer, 96-2:214-217, 1974*. We prove here that there exists a unique non-negative analytic solution for small positive values of the parameter on which the problem depends.

Key words Modified error function, error function, phase change problem, temperature-dependent thermal conductivity, nonlinear second order ordinary differential equation.

2000 MSC 35R35, 80A22, 34B15, 34B08.

1 Introduction

In 1974, Cho and Sunderland [2] studied a solidification process with temperature-dependent thermal conductivity and obtained an explicit similarity solution in terms of what they called a *modified error*

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function. This function is defined as the solution to the following non-linear differential problem:

$$[(1 + \delta y(x))y'(x)]' + 2xy'(x) = 0 \quad 0 < x < +\infty \quad (1a)$$

$$y(0) = 0 \quad (1b)$$

$$y(+\infty) = 1 \quad (1c)$$

where $\delta \geq -1$ is given. Graphics for numerical solutions of (1) for different values of δ can be found in [2]. The classical error function is defined by:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-z^2) dz, \quad x > 0, \quad (2)$$

and it is a solution to (1) when $\delta = 0$. This makes meaningful the denomination *modified error function* given for the solution to problem (1).

The modified error function has also appeared in the context of diffusion problems before 1974 [4, 12]. It was also used later in several opportunities to find similarity solutions to phase-change processes [1, 5–8, 10]. It was cited in [3], where several non-linear ordinary differential problems arise from a wide variety of fields are presented. Closed analytical solutions for Stefan problems with variable diffusivity is given in [11]. Temperature-dependent thermal coefficients are very important in thermal analysis, e.g. see [9]. Nevertheless, to the knowledge of the authors, the existence and uniqueness of the solution to problem (1) has not been yet proved. This article is devoted to prove it for small $\delta > 0$ using a fixed point strategy.

2 Existence and uniqueness of solution to problem (1)

The main idea developed in this Section is to study problem (1) through the linear problem given by the differential equation:

$$[(1 + \delta \Psi_h(x))y'(x)]' + 2xy'(x) = 0, \quad 0 < x < +\infty, \quad (1a^*)$$

and conditions (1b), (1c). The function Ψ_h in (1a^{*}) is defined by:

$$\Psi_h(x) = 1 + \delta h(x), \quad x > 0, \quad (3)$$

where $\delta > 0$, $h \in K \subset X$ is given and:

$$X = \{h : \mathbb{R}_0^+ \rightarrow \mathbb{R} / h \text{ is an analytic function, } \|h\|_\infty < \infty\} \quad (4a)$$

$$K = \{h \in X / \|h\|_\infty \leq 1, 0 \leq h, h(0) = 0, h(+\infty) = 1\}. \quad (4b)$$

Hereinafter, we will refer to the problem given by (1a^{*}), (1b) and (1c) as problem (1^{*}). Let us observe that K is non-empty closed subset of the Banach space X .

The advantage in considering the linear equation (1a^{*}) is that it can be easily solved through the substitution $v = y'$. Thus, we have the following result:

Theorem 2.1. *Let $h \in K$ and $\delta > 0$. The solution y to problem (1^{*}) is given by:*

$$y(x) = C_h \int_0^x \frac{1}{\Psi_h(\eta)} \exp \left(-2 \int_0^\eta \frac{\xi}{\Psi_h(\xi)} d\xi \right) d\eta \quad x \geq 0, \quad (5)$$

where the constant C_h is defined by:

$$C_h = \left(\int_0^{+\infty} \frac{1}{\Psi_h(\eta)} \exp \left(-2 \int_0^\eta \frac{\xi}{\Psi_h(\xi)} d\xi \right) d\eta \right)^{-1}. \quad (6)$$

Proof. Let us first observe that the constant C_h given by (6) is well defined, that is, that $C_h \in \mathbb{R}$. In fact, we have:

$$\begin{aligned} |C_h^{-1}| &= \int_0^{+\infty} \frac{1}{\Psi_h(\eta)} \exp \left(-2 \int_0^\eta \frac{\xi}{\Psi(\xi)} d\xi \right) d\eta \\ &\geq \frac{1}{1+\delta} \int_0^{+\infty} \exp(-\eta^2) d\eta = \frac{\sqrt{\pi}}{2(1+\delta)} \end{aligned} \quad (7)$$

Now the proof follows easily by checking that the function y given by (5) satisfies problem (1^{*}). ■

The following result is an immediate consequence of Theorem 2.1.

Corollary 2.1. *Let $y \in K$ and $\delta > 0$. Then y is a solution to problem (1) if and only if y is a fixed point of the operator τ from K to X defined by:*

$$\tau(h)(x) = C_h \int_0^x \frac{1}{\Psi_h(\eta)} \exp \left(-2 \int_0^\eta \frac{\xi}{\Psi_h(\xi)} d\xi \right) d\eta \quad x > 0, \quad (8)$$

with C_h given by (6).

Remark 1. Observe that $\tau(K) \subset K$.

We will now focus on analyzing when τ has only one fixed point. The estimations summarized next will be useful in the following.

Lemma 2.1. *Let $h, h_1, h_2 \in K$, $\delta > 0$ and $x \geq 0$. We have:*

$$a) \int_0^x \left| \frac{\exp \left(-2 \int_0^\eta \frac{\xi}{\Psi_{h_1}(\xi)} d\xi \right)}{\Psi_{h_1}(\eta)} - \frac{\exp \left(-2 \int_0^\eta \frac{\xi}{\Psi_{h_2}(\xi)} d\xi \right)}{\Psi_{h_2}(\eta)} \right| d\eta$$

$$\leq \frac{\sqrt{\pi}}{4} \delta \sqrt{1+\delta} (3+\delta) \|h_1 - h_2\|_\infty,$$

$$b) |C_{h_1} - C_{h_2}| \leq \frac{1}{\sqrt{\pi}} \delta \sqrt{1+\delta} (1+\delta)^2 (3+\delta) \|h_1 - h_2\|_\infty,$$

$$c) \int_0^x \frac{1}{\Psi_h(\eta)} \exp\left(-2 \int_0^\eta \frac{\xi}{\Psi_h(\xi)} d\xi\right) d\eta \leq \frac{\sqrt{\pi(1+\delta)}}{2}.$$

Proof. Let f be the real function defined on \mathbb{R}_0^+ by $f(x) = \exp(-2x)$. If $h_1 \leq h_2$, it follows from the Mean Value Theorem applied to function f that:

$$\begin{aligned} & \left| \exp\left(-2 \int_0^\eta \frac{\xi}{\Psi_{h_1}(\xi)} d\xi\right) - \exp\left(-2 \int_0^\eta \frac{\xi}{\Psi_{h_2}(\xi)} d\xi\right) \right| \\ &= 2 \exp\left(-2 \int_0^\eta \frac{\xi}{\Psi_{h_3}(\xi)} d\xi\right) \left| \int_0^\eta \frac{\xi}{\Psi_{h_1}(\xi)} d\xi - \int_0^\eta \frac{\xi}{\Psi_{h_2}(\xi)} d\xi \right| \\ &\leq \delta \|h_2 - h_1\|_\infty \eta^2 \exp\left(\frac{-\eta^2}{1+\delta}\right), \end{aligned} \tag{9}$$

where $h_1 \leq h_3 \leq h_2$. Now a) follows from regular computations. When $h_1 \not\leq h_2$, as the LHS in a) can be bounded for the same expression but applied to $h_m = \min\{h_1, h_2\}$ and $h_M = \max\{h_1, h_2\}$, the proof runs as before and it is completed having into consideration that $\|h_1 - h_2\|_\infty = \|h_M - h_m\|_\infty$.

The proof of b) follows from a), and c) can be obtained from regular computations. ■

Theorem 2.2. Let $\delta_1 > 0$ be the only one positive solution to the equation:

$$\frac{x}{2} (1+x)^{3/2} (3+x) [1 + (1+x)^{3/2}] = 1. \tag{10}$$

If $0 < \delta < \delta_1$, then τ is a contraction.

Proof. Let g be the real function defined by:

$$g(x) = \frac{x}{2} (1+x)^{3/2} (3+x) [1 + (1+x)^{3/2}] \quad x \geq 0. \tag{11}$$

Since g is an increasing function from 0 to $+\infty$, we have that equation (10) admits only one positive solution δ_1 .

Let be now $h_1, h_2 \in K$ and $x \geq 0$. From Lemma 2.1, (7) and:

$$\begin{aligned} & |\tau(h_1)(x) - \tau(h_2)(x)| \\ &\leq C_{h_1} \int_0^x \left| \frac{\exp\left(-2 \int_0^\eta \frac{\xi}{\Psi_{h_1}(\xi)} d\xi\right)}{\Psi_{h_1}(\eta)} - \frac{\exp\left(-2 \int_0^\eta \frac{\xi}{\Psi_{h_2}(\xi)} d\xi\right)}{\Psi_{h_2}(\eta)} \right| d\eta \\ &+ |C_{h_1} - C_{h_2}| \int_0^x \frac{1}{\Psi_{h_2}(\eta)} \exp\left(-2 \int_0^\eta \frac{\xi}{\Psi_{h_2}(\xi)} d\xi\right) d\eta, \end{aligned}$$

it follows that $||\tau(h_1) - \tau(h_2)||_\infty \leq \gamma ||h_1 - h_2||_\infty$, where $\gamma = g(\delta)$. Recalling that g is an increasing function, it follows that τ is a contraction when $0 < \delta < \delta_1$. ■

Remark 2. From a numerical computation, it can be found that:

$$0.203701 < \delta_1 < 0.203702.$$

We are now in the position to formulate our main result:

Corollary 2.2. *Let δ_1 be as in Theorem 2.2. If $0 < \delta < \delta_1$, then problem (1) has a unique non-negative analytic solution.*

Proof. It is a direct consequence of Corollary 2.1, Theorem 2.2 and the Banach Fixed Point Theorem. ■

Acknowledgments

This paper has been partially sponsored by the Project PIP No. 0534 from CONICET-UA (Rosario, Argentina) and AFOSR-SOARD Grant FA 9550-14-1-0122.

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